Communication for maths

Term 2 week 4 - Integration: On integration by-parts and by substitution

Introduction

- These slides illustrate certain aspects relating to the presentation of curve sketching.
- Some relate to the presentation of all maths in general and some are specific to the presentation of curve sketching.
- Not all aspects of mathematics communication that we have studied so far will be presented here.
- Revise the slides on the previous topics where appropriate.

Layout / spacing

Example 1

Question: Find $\int (3x^2+3)\sin(x^3+3)\,dx$

Incorrect Solution

$$u = x^3 + \underbrace{3x \ du}_{\text{No}} = 3x^2 + 3x\,dx$$

Layout / spacing

Example 2

Q: Find $\int x \cdot \sqrt{x-3}\ dx$

Sol let $u = x - 3$

$\therefore \quad u + 3 = x$

$du = dx$

So $I = \int (u-3)\sqrt{u} \qquad du = \int u^{3/2} - 3u^{1/2}$

du

No

Do not mix your variables

Example:

State an integral in terms of x or u only but not both

Q: Find $\int x \cdot \sqrt{x-3}\, dx$

Sol: Let $I = \int x \cdot \sqrt{x-3}\, dx$

Then $u = x - 3$, $du = dx$

So

$$I = \int x \sqrt{u}\, du$$ No

But $u + 3 = x$

$\therefore$ $I = \int (u+3) \sqrt{u}\, du$

Do not mix your variables

Example:

Show all necessary working before you write the transformed integral

Q: Find $I = \int x.\sqrt{x-3}\, dx$

Sol: let $u = x-3$;

$\therefore du = x\, dx$

& $u+3 = x$

Yes

$\therefore I = \int (u+3).\sqrt{u}\, du$ Yes

No free-standing expressions

Example

Question: Find $\int (3x^2+3)\sin(x^3+3)\,dx$

Incorrect Solution

$u = x^3 + 3x \quad du = 3x^2 + 3x\,dx$ ← **No**

$\int \sin u\, du$

$\therefore \quad -\cos u + C$ ← **No**

Proofs: Use "Claim – Proof – QED"

Example: Incorrect solution. Why?

Question: Prove $\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \frac{du}{dx}\, dx$

Incorrect Solution

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\therefore \quad y = \int u \cdot \frac{dv}{dx}\, dx + \int v \frac{du}{dx}\, dx$$

So $$\int u \cdot \frac{dv}{dx}\, dx = uv - \int v \cdot \frac{du}{dx}\, dx$$

Proofs: Use “Claim – Proof – QED”

Example: Correct solution

Use of “dx” in an integral

Example

Incorrect

solution.

Why?

Question : Find $\int x . e^x \, dx$

Incorrect Solution

$$u = x \longrightarrow du = dx$$

$$\frac{dv}{dx} = e^x \longrightarrow v = e^x$$

So $\quad xe^x - \int e^x \quad \longleftarrow$ **No**

Use of constant of integration

Example

Incorrect

solution.

Why?

Question : Find $\int x . e^x dx$

Incorrect Solution

$u = x \longrightarrow du = dx$

$\frac{dv}{dx} = e^x \longrightarrow v = e^x$

So $xe^x - \int e^x$

$= xe^x - e^x$ ⟵ **No**

Correct use of notation

Example

Find $\int x \cdot e^x \, dx$

Sol : ....

$u = x \longrightarrow du = dx$

$dv = e^x \underbrace{\longrightarrow}_{\text{No}} v = e^x$

....

Justification

By-parts: Explicitly write

$$u = \dots\ ,\ du = \dots\ dx,\ dv = \dots\ ,\ v = \dots$$

<u>Sol</u> : Let $I = \int x \cdot e^x \, dx$

$\therefore I = xe^x - \int e^x \, dx$ ⟵ **No**

Justification

By-parts: Explicitly write

$$u = \ldots, \; du = \ldots dx, \; dv = \ldots, \; v = \ldots$$

<u>Sol</u>: Let $I = \int x \cdot e^x \, dx$

$\therefore \; u = \ldots, \quad du = \ldots dx$

$dv = \ldots dx, \quad v = \ldots$

Yes

So $I = \ldots - \int \ldots dx$

Limits of integration under a change of variable

Example

<u>Question</u>: Find $\int_1^2 x(x^2-1)\,dx$

<u>Incorrect Solution</u>:

let $u = x^2 - 1$; $\therefore du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

So $\int_1^2 x(x^2-1)\,dx = \underbrace{\int_1^2}_{\text{No}} \frac{1}{2}u\,du$

Limits of integration under a change of variable

Example

<u>Correct Solution</u>

Let $u = x^2 - 1$; Then $du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$

And if $x = 1$, $u = 0$
if $x = 2$, $u = 3$

Yes

$$\therefore \int_1^2 x(x^2-1)\,dx = \underbrace{\int_0^3}_{\text{Yes}} \frac{1}{2} u\,du = \left[\frac{1}{4}u^2\right]_0^3$$

Limits of integration under a change of variable

- **Summary**
  - Write all integrals in terms of one variable only, i.e.

$$\int_a^b f(x)\ dx \qquad \text{or} \qquad \int_c^d g(u)\ du$$

(1) (2)

  - Do all necessary change-of-variable algebra before you write (2):
    - u = …; du = …dx;
    - x = a ⇒ u = c; x = b ⇒ u = d;

Limits of integration for integration by parts

- **Example**
- *(*definite integration by parts example*

RHS is uv – int (v du). Show example where limits are included on the "int" but not on the "uv" bit)*

On stating integrals

Example

Question: Find $\int_1^2 x(x^2-1)\,dx$

Incorrect Solution:

let $u = x^2 - 1$; $\therefore \frac{1}{2}du = x\,dx$

So

$$\int_1^2 x(x^2-1)\,dx = \frac{1}{2}\int u\,du$$

No

On stating integrals

- **Explanation**

  Don't mix definite integrals with indefinite integrals on the same step;

  If you want to write in terms of indefinite integrals, rewrite the question.

  See right for example.

Let

$$I = \int x(x^2+1)\,dx$$

$$\cdots$$

$$\cdots$$

$$\int x(x^2+1)\,dx = \int \frac{1}{2} u\,du$$

So

$$\int_1^2 x(x^2+1)\,dx = \int_0^3 \frac{1}{2} u\,du$$

$$\cdots$$

Exercise

Identify and correct all errors of presentation in the exercise handed out.

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Appendix

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Correct use of symbols: Limits of integration

Example

Correct

$$\int_n^{n+1} \frac{1}{x}\,dx = \left[\ln x\right]_n^{n+1} = \ln(n+1) - \ln n$$

Incorrect

“Perform” versus “Evaluate”

Examples: Perform/find vs evaluate/calculate:

Yes	No
We perform $\int e^x \, dx$	We evaluate $\int e^x \, dx$
We evaluate $\int_1^2 e^x \, dx$	We perform $\int_1^2 e^x \, dx$

“Perform” versus “Evaluate”

Explanation

- “Perform the integral” or “Find the integral”: Used for indefinite integrals to find an algebraic expression.

- “Evaluate the integral” or “Calculate the integral”: Used for definite integrals to get a numerical answer.